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Tensor Products and Probability Weights

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We study a general tensor product for two collections of related physical operations or observations. This is a free product, subject only to the condition that the operations in the first collection fail to have any influence on the statistics of operations in the second collection and vice versa. In the finite-dimensional case, it is shown that the vector space generated by the probability weights on the general tensor product is the algebraic tensor product of the vector spaces generated by the probability weights on the components. The relationship between the general tensor product and the tensor product of Hilbert spaces is examined in the light of this result.

1. BACKGROUND

According to von Neumann, a quantum mechanical system $\mathcal{S}$ is represented by a complex, separable Hilbert space $\mathcal{H}$ in such a way that observables on $\mathcal{S}$ correspond to self-adjoint operators on $\mathcal{H}$, and states on $\mathcal{S}$ are represented by density operators (positive, self-adjoint, trace-class operators with trace 1) on $\mathcal{H}$. If A is the self-adjoint operator on $\mathcal{H}$ corresponding to the observable $\mathcal{O}$ on $\mathcal{S}$, if $M \mapsto P_M$ is the spectral measure for A, and if W is the density operator representing the state of $\mathcal{S}$, then the probability that a measurement of $\mathcal{O}$ will yield a numerical result in the Borel set $M \subseteq \mathbb{R}$ is given by trace(WP_M). This suggests that, in general, a projection operator P (self-adjoint idempotent) on $\mathcal{H}$ can be regarded as a yes/no proposition about $\mathcal{S}$ in such a way that the probability in state W that the answer to P is "yes" is given by trace(WP) (von Neumann, 1955, p. 247; Birkhoff and von Neumann, 1936).

The set $L(\mathcal{H})$ of all projection operators P on $\mathcal{H}$ forms a complete orthomodular lattice (Kalmbach, 1983). Since the projection operators $P \in L(\mathcal{H})$ can be regarded as yes/no propositions about $\mathcal{S}$, it is natural to

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refer to the orthomodular lattice $L(\mathscr{H})$ as the "logic" of $\mathscr{S}$ (Birkhoff and von Neumann, 1936). Each density operator W on $\mathscr{H}$ induces a probability measure μ_W: $L(\mathscr{H}) \to [0, 1] \subseteq \mathbb{R}$ such that

$$\mu_W(P) = \text{trace}(WP) \qquad \text{for all} \quad P \in L(\mathscr{H})$$

By a celebrated theorem of Gleason (1957), if $\dim(\mathscr{H}) \geq 3$, then the mapping $W \mapsto \mu_W$ is a bijection from the set $\mathscr{W}(\mathscr{H})$ of all density operators on $\mathscr{H}$ onto the set of all probability measures on $L(\mathscr{H})$. The convex set $\mathscr{W}(\mathscr{H})$ forms a cone base for its linear span, the base-norm space $\mathscr{V}(\mathscr{H})$ of all self-adjoint trace-class operators on $\mathscr{H}$, and the base norm on $\mathscr{V}(\mathscr{H})$ coincides with the trace norm (Rüttimann, 1985b). Each element $A \in \mathscr{V}(\mathscr{H})$ induces a (signed) measure μ_A: $L(\mathscr{H}) \to \mathbb{R}$ such that

$$\mu_A(P) = \text{trace}(WP) \qquad \text{for all} \quad P \in L(\mathscr{H})$$

A normalized vector ψ in $\mathscr{H}$ determines a projection operator P_ψ according to the formula

$$P_\psi(\phi) = \langle \phi, \psi \rangle \psi \qquad \text{for all} \quad \phi \in \mathscr{H}$$

(We use the mathematician's convention that the inner product is conjugate linear in its *second* argument.) Thus, a maximal orthonormal set E of vectors in $\mathscr{H}$ determines a maximal orthonormal set $\{P_\psi | \psi \in E\}$ of propositions in $L(\mathscr{H})$, and therefore can be construed as representing a maximal observation in the sense of Dirac (1930; Foulis and Randall, 1985).

When two quantum mechanical systems $\mathscr{S}_1$ and $\mathscr{S}_2$, represented by Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, are "coupled" to form a composite system $\mathscr{S}_1 + \mathscr{S}_2$, it has been supposed that the Hilbert space representing $\mathscr{S}_1 + \mathscr{S}_2$ ought to be the Hilbert-space tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ (Jauch, 1968). [In spite of the fact that Schrödinger was in part responsible for this representation, he was uneasy about some of its consequences (Schrödinger, 1935, 1936).] In the tensor product representation, the states for $\mathscr{S}_1 + \mathscr{S}_2$ correspond to density operators $W \in \mathscr{W}(\mathscr{H}_1 \otimes \mathscr{H}_2)$. For such a W, the reduced states, or marginals, $W_1 \in \mathscr{W}(\mathscr{H}_1)$ and $W_2 \in \mathscr{W}(\mathscr{H}_2)$ affiliated with the component systems $\mathscr{S}_1$ and $\mathscr{S}_2$ are uniquely determined by the conditions

$$\text{trace}(W_1 P) = \text{trace}[W(P \otimes \mathbb{1})]$$

$$\text{trace}(W_2 Q) = \text{trace}[W(\mathbb{1} \otimes Q)]$$

for all $P \in L(\mathscr{H}_1)$ and all $Q \in L(\mathscr{H}_2)$.

Now suppose that the system $\mathscr{S}_1 + \mathscr{S}_2$ is represented by the Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. If E is a maximal orthogonal set of vectors in $\mathscr{H}_1$, we can define the *conditional probability of $Q \in L(\mathscr{H}_2)$ in state W, given that the maximal observation E was made on $\mathscr{S}_1$*, to be

$$_E\mu_W(Q) = \sum_{\psi \in E} \text{trace}[W(P_\psi \otimes Q)]$$

In the sense of Dirac (1930, p. 13), we say that the system $\mathcal{S}_1$ *has no influence* on the system $\mathcal{S}_2$ if, for every state $W \in \mathcal{W}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the conditional probability measure ${}_E\mu_W$ on $L(\mathcal{H}_2)$ is independent of the choice of the maximal observation E on $\mathcal{H}_1$. However, it is easy to see that

$$ {}_E\mu_W(Q) = \operatorname{trace}(W_2 Q) $$

is automatically independent of the choice of E; hence, *if $\mathcal{S}_1 + \mathcal{S}_2$ is represented by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, then $\mathcal{S}_1$ has no influence on $\mathcal{S}_2$*. Likewise, $\mathcal{S}_2$ has no influence on $\mathcal{S}_1$. This is a feature of the tensor product representation of coupled systems that is often overlooked by those who study the mathematical foundations of quantum mechanics: The representation of coupled systems by the Hilbert-space tensor product allows only *correlations* between the systems—there can be no *influence* of one system upon the other!

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional, a simple dimension argument shows that there is a natural vector-space isomorphism

$$ \mathcal{V}(\mathcal{H}_1) \otimes \mathcal{V}(\mathcal{H}_2) \simeq \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2) $$

A general tensor product for two collections of related physical operations or observations was introduced in Foulis and Randall (1980) and Randall and Foulis (1980). This general tensor product is a "free product," subject only to the condition that the operations in the first collection fail to have any influence (in the sense of Dirac) on the statistics of operations in the second collection and vice versa. The main theorem of the present paper is an extension of the isomorphism above to this generalized tensor product.

In order to set the stage for the subsequent developments, we consider a separable Hilbert space $\mathcal{H}$ over either of the fields $\mathbb{R}$ or $\mathbb{C}$. We define $\mathcal{A}$ to be the set of all maximal orthonormal sets of vectors E in $\mathcal{H}$, and we think of such an E as a maximal observation in the sense indicated above. Then $X = \bigcup \{E \mid E \in \mathcal{A}\}$ is the unit sphere in $\mathcal{H}$, and the probability measures μ on the projection lattice $L(\mathcal{H})$ are in a natural bijective correspondence $\mu \mapsto \omega$ with the mappings $\omega\colon X \to [0, 1] \subseteq \mathbb{R}$ satisfying the condition

$$ \sum_{\psi \in E} \omega(\psi) = 1 \qquad \text{for all} \quad E \in \mathcal{A} $$

in such a way that, for all $\psi \in X$, $\omega(\psi) = \mu(P_\psi)$. (If E is infinite, the sum is understood in the sense of unordered summability.) Such a mapping ω is called a *probability weight* on $\mathcal{A}$.

Denote by $\Omega(\mathcal{A})$ the convex subset of the real vector space $\mathbb{R}^X$ consisting of all of the probability weights ω on $\mathcal{A}$. The linear span of $\Omega(\mathcal{A})$ in $\mathbb{R}^X$ forms a base-norm space $V(\mathcal{A})$ with $\Omega(\mathcal{A})$ as cone base (Cook, 1985). If $\dim(\mathcal{H}) \geq 3$, then the correspondence

$$ A \mapsto \mu_A = \mu \mapsto \omega $$

establishes an affine isomorphism of $\mathscr{W}(\mathscr{H})$ onto $\Omega(\mathscr{A})$, and this isomorphism has a unique extension to a linear isomorphism

$$\mathscr{V}(\mathscr{H}) \simeq V(\mathscr{A})$$

Thus, the set of sets $\mathscr{A}$, divested of all the remaining structure of the Hilbert space $\mathscr{H}$, serves as a carrier of the measures on $L(\mathscr{H})$ induced by the self-adjoint trace-class operators on $\mathscr{H}$.

2. PROBABILITY WEIGHTS

In the following definitions, we generalize the ideas introduced at the end of the last section. We begin by considering an arbitrary nonempty set $\mathscr{A}$ of nonempty sets

$$\mathscr{A} = \{E, F, G, \ldots\}$$

Such a set of sets, with or without additional requirements, has been called a *space* (Greechie and Miller, 1970), a *hypergraph* (Gudder *et al.*, 1987), or a *cover space* (Gudder, 1986). In the operational approach to the mathematical foundations of quantum mechanics, such an $\mathscr{A}$ (subject to suitable regularity conditions) is regarded as representing a catalogue, or *manual*, of operations or experiments (Randall and Foulis, 1973, 1978, 1985; Foulis *et al.*, 1983). This brings us to the following definition:

Definition 2.1. A *quasimanual* is a nonempty set $\mathscr{A}$ of nonempty sets satisfying the condition that

$$E, F \in \mathscr{A} \qquad \text{with} \quad E \subseteq F \Rightarrow E = F$$

If $\mathscr{A}$ is a quasimanual, we use the notation $\bigcup \mathscr{A}$ for the union of all of the sets in $\mathscr{A}$, so that

$$\bigcup \mathscr{A} = \bigcup \{E \mid E \in \mathscr{A}\}$$

The set $\mathscr{A}$ of all maximal orthonormal subsets of a Hilbert space $\mathscr{H}$ may be regarded as one of the prototypical examples of a quasimanual. In this case, $\bigcup \mathscr{A}$ is the unit sphere in $\mathscr{H}$. Another important example is obtained as follows: Let S be a nonempty set, let $\mathscr{M}$ be a σ-field of subsets of S, and consider the Borel space $(S, \mathscr{M})$. Then $\mathscr{B}(S, \mathscr{M})$, the set of all countable partitions of S into nonempty disjoint sets in $\mathscr{M}$, is a quasimanual and $\bigcup \mathscr{B}(S, \mathscr{M}) = \mathscr{M} \backslash \{\varnothing\}$.

Definition 2.2 Let $\mathscr{A}$ be a quasimanual with $X = \bigcup \mathscr{A}$. A *probability weight* on $\mathscr{A}$ is a mapping $\omega\colon X \to [0, 1] \subseteq \mathbb{R}$ such that

$$\sum_{x \in E} \omega(x) = 1 \qquad \text{for all} \quad E \in \mathscr{A}$$

We denote by $\Omega(\mathscr{A})$ the set of all probability weights on $\mathscr{A}$. For $\omega \in \Omega(\mathscr{A})$, we define the *support* of ω, in symbols $\mathrm{supp}(\omega)$, by $\mathrm{supp}(\omega) = \{x \in X | \omega(x) \neq 0\}$. If $A \subseteq E \in \mathscr{A}$ and $\omega \in \Omega(\mathscr{A})$, we define

$$\omega(A) = \sum_{x \in A} \omega(x)$$

The linear span of $\Omega(\mathscr{A})$ in the real vector space $\mathbb{R}^X$ is denoted by $V(\mathscr{A})$.

For a Borel space $(S, \mathcal{M})$, we observe that $V(\mathscr{B}(S, \mathcal{M}))$ is naturally isomorphic to the space of all (signed) measures of bounded variation on $(S, \mathcal{M})$ in such a way that $\Omega(\mathscr{B}(S, \mathcal{M}))$ corresponds to the space of all probability measures on $(S, \mathcal{M})$.

Theorem 2.3. If $\mathscr{A}$ is a quasimanual and $\Omega(\mathscr{A}) \neq \varnothing$, then $(V(\mathscr{A}), \Omega(\mathscr{A}))$ is a base-norm space and, under the base norm, $V(\mathscr{A})$ is a Banach space.

Proof. Cook (1985).

In what follows, $V(\mathscr{A})^*$ denotes the Banach dual of $V(\mathscr{A})$ and $e \in V(\mathscr{A})^*$ denotes the dual order unit. Note that, for any $E \in \mathscr{A}$ and any $\nu \in V(\mathscr{A})$, we have

$$e(\nu) = \sum_{x \in E} \nu(x)$$

Definition 2.4. Let $\mathscr{A}$ be a quasimanual such that $\Omega(\mathscr{A}) \neq \varnothing$ and let $X = \bigcup \mathscr{A}$. For each $x \in X$, define the *frequency functional* $f_x \in V(\mathscr{A})^*$ by $f_x(\nu) = \nu(x)$ for all $\nu \in V(\mathscr{A})$. A subset B of X is said to be *$\mathscr{A}$-basic* if $\{f_x | x \in B\} \cup \{e\}$ is a total subset of $V(\mathscr{A})^*$.

Note that each frequency functional f_x belongs to the dual order interval $[0, e]$; hence, it is a *counter* in the sense of Rüttimann (1985a). If $V(\mathscr{A})$ is finite-dimensional and B is an $\mathscr{A}$-basic subset of $X = \bigcup \mathscr{A}$, then we have

$$\dim(V(\mathscr{A})) = \dim(V(\mathscr{A})^*) \leq \# B + 1$$

where we denote by $\# B$ the cardinal number of the set B. The following lemma is a consequence of the fact that, for $\omega, \nu \in \Omega(\mathscr{A})$, we have $\omega - \nu \in \ker(e)$.

Lemma 2.5. Let $\mathscr{A}$ be a quasimanual such that $\Omega(\mathscr{A}) \neq \varnothing$ and let $B \subseteq X = \bigcup \mathscr{A}$. Then B is $\mathscr{A}$-basic if and only if, for $\omega, \eta \in \Omega(\mathscr{A})$, $\omega|_B = \eta|_B \Rightarrow \omega = \eta$.

Notation 2.6. In what follows, we use the notation xy for the ordered pair (x, y). Likewise, Ey denotes the set of all ordered pairs (x, y) with $x \in E$, xF denotes the set of all ordered pairs (x, y) with $y \in F$, and EF denotes the set of all ordered pairs (x, y) with $x \in E$ and $y \in F$.

For the remainder of this section, we assume that $\mathscr{A}$ and $\mathscr{B}$ are quasimanuals with $X = \bigcup \mathscr{A}$ and $Y = \bigcup \mathscr{B}$.

Definition 2.7. We define the *Cartesian product* $\mathscr{A} \times \mathscr{B}$ of the quasimanuals $\mathscr{A}$ and $\mathscr{B}$ by $\mathscr{A} \times \mathscr{B} = \{EF | E \in \mathscr{A}, F \in \mathscr{B}\}$. If $\omega \in \Omega(\mathscr{A} \times \mathscr{B})$, $E \in \mathscr{A}$, and $F \in \mathscr{B}$, we define ${}_E\omega \in \Omega(\mathscr{B})$ and $\omega_F \in \Omega(\mathscr{A})$ by ${}_E\omega(y) = \omega(Ey)$ for all $y \in Y$ and $\omega_F(x) = \omega(xF)$ for all $x \in X$.

Note that $XY = \bigcup (\mathscr{A} \times \mathscr{B})$.

Definition 2.8. If $\omega \in \Omega(\mathscr{A} \times \mathscr{B})$, then we say that ω *exhibits no influence of $\mathscr{A}$ on $\mathscr{B}$* if ${}_E\omega \in \Omega(\mathscr{B})$ is independent of the choice of $E \in \mathscr{A}$. Likewise, we say that ω *exhibits no influence of $\mathscr{B}$ on $\mathscr{A}$* if $\omega_F \in \Omega(\mathscr{A})$ is independent of the choice of $F \in \mathscr{B}$.

By adjoining subsets of $X = \bigcup \mathscr{A}$ to a quasimanual $\mathscr{A}$ in such a way as to preserve the condition in Definition 2.1, we obtain a larger quasimanual $\mathscr{A}'$ with $X = \bigcup \mathscr{A}'$. Obviously, $\Omega(\mathscr{A}') \subseteq \Omega(\mathscr{A})$. Thus, the passage from $\mathscr{A}$ to a larger $\mathscr{A}'$ (with the same union X) can be regarded as one means for imposing a condition on the probability weights $\omega \in \Omega(\mathscr{A})$, i.e., ω satisfies the condition in question if and only if ω belongs to the smaller set $\Omega(\mathscr{A}')$. In the next definition, we introduce an enlargement of $\mathscr{A} \times \mathscr{B}$ that has the effect of imposing the condition that $\mathscr{B}$ has no influence on $\mathscr{A}$ (see Theorem 2.10).

Definition 2.9. We define the *forward operational product* $\overrightarrow{\mathscr{A}\mathscr{B}}$ of the quasimanuals $\mathscr{A}$ and $\mathscr{B}$ by

$$\overrightarrow{\mathscr{A}\mathscr{B}} = \left\{ \bigcup_{x \in E} xF_x \,\middle|\, E \in \mathscr{A} \text{ and } F_x \in \mathscr{B} \text{ for every } x \in E \right\}$$

Note that $\mathscr{A} \times \mathscr{B} \subseteq \overrightarrow{\mathscr{A}\mathscr{B}}$ and $XY = \bigcup (\mathscr{A} \times \mathscr{B}) = \bigcup \overrightarrow{\mathscr{A}\mathscr{B}}$. Therefore, the passage from the Cartesian product to the forward operational product can be regarded as imposing a condition on the probability weights in $\Omega(\mathscr{A} \times \mathscr{B})$. The following theorem shows just what this condition is.

Theorem 2.10. If $\omega \in \Omega(\mathscr{A} \times \mathscr{B})$, then $\omega \in \Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ if and only if ω exhibits no influence of $\mathscr{B}$ on $\mathscr{A}$.

Proof. Randall and Foulis (1980).

The arrow in the notation $\overrightarrow{\mathscr{A}\mathscr{B}}$ is supposed to indicate the *direction of influence* (if any) from $\mathscr{A}$ to $\mathscr{B}$. By reversing all ordered pairs in the definition of the forward operational product, we obtain the *backward operational product,* in which the influence (if any) is from $\mathscr{B}$ to $\mathscr{A}$.

Definition 2.11. Define the "switching mapping" π: $YX \to XY$ by $\pi(yx) = xy$ for $x \in X$, $y \in Y$. Then, the *backward operational product* $\overleftarrow{\mathscr{A}\mathscr{B}}$ is defined by $\overleftarrow{\mathscr{A}\mathscr{B}} = \{\pi(G) | G \in \overrightarrow{\mathscr{B}\mathscr{A}}\}$.

Corollary 2.12. If $\omega \in \Omega(\mathcal{A} \times \mathcal{B})$, then $\omega \in \Omega(\overleftarrow{\mathcal{A}\mathcal{B}})$ if and only if ω exhibits no influence of $\mathcal{A}$ on $\mathcal{B}$.

Proof. Randall and Foulis (1980).

In what follows, we denote generic probability weights on $\overrightarrow{\mathcal{A}\mathcal{B}}$ and $\overleftarrow{\mathcal{A}\mathcal{B}}$ by $\vec{\omega}$ and $\overleftarrow{\omega}$, respectively. As a consequence of Theorem 2.10 and Corollary 2.12, we can make the following definition.

Definition 2.13. Let $\vec{\omega} \in \Omega(\overrightarrow{\mathcal{A}\mathcal{B}})$ and $\overleftarrow{\omega} \in \Omega(\overleftarrow{\mathcal{A}\mathcal{B}})$. Define the *reduced,* or *marginal,* weights $\vec{\omega}_{\mathcal{B}} \in \Omega(\mathcal{A})$ and ${}_{\mathcal{A}}\overleftarrow{\omega} \in \Omega(\mathcal{B})$ by $\vec{\omega}_{\mathcal{B}} = \vec{\omega}_F$ for any $F \in \mathcal{B}$ and ${}_{\mathcal{A}}\overleftarrow{\omega} = {}_E\overleftarrow{\omega}$ for any $E \in \mathcal{A}$. For $x \in \operatorname{supp}(\vec{\omega}_{\mathcal{B}})$, define ${}_x\vec{\omega}$: $Y \to \mathbb{R}$ by ${}_x\vec{\omega}(y) = \vec{\omega}(xy)/\vec{\omega}_{\mathcal{B}}(x)$ for all $y \in Y$. Likewise, for $y \in \operatorname{supp}({}_{\mathcal{A}}\overleftarrow{\omega})$, define $\overleftarrow{\omega}_y$: $X \to \mathbb{R}$ by $\overleftarrow{\omega}_y(x) = \overleftarrow{\omega}(xy)/{}_{\mathcal{A}}\overleftarrow{\omega}(y)$ for all $x \in X$.

We omit the straightforward proof of the following lemma:

Lemma 2.14. Let $\vec{\omega} \in \Omega(\overrightarrow{\mathcal{A}\mathcal{B}})$ and $\overleftarrow{\omega} \in \Omega(\overleftarrow{\mathcal{A}\mathcal{B}})$. Let $x \in X$ and $y \in Y$. Then ${}_x\vec{\omega} \in \Omega(\mathcal{B})$ and, likewise, $\overleftarrow{\omega}_y \in \Omega(\mathcal{A})$. Furthermore, if $\vec{\omega}_{\mathcal{B}}(x) = 0$, then $\vec{\omega}(xy) = 0$ for all $y \in Y$, and, if ${}_{\mathcal{A}}\overleftarrow{\omega}(y) = 0$, then $\overleftarrow{\omega}(xy) = 0$ for all $x \in X$.

The probability weights ${}_x\vec{\omega}$ and $\overleftarrow{\omega}_y$ in Lemma 2.14 are referred to as $\vec{\omega}$ *preconditioned by* x and $\overleftarrow{\omega}$ *postconditioned by* y, respectively. With the notation of Lemma 2.14, we have for every $x \in X$ and every $y \in Y$,

$$\vec{\omega}(xy) = \begin{cases} \vec{\omega}_{\mathcal{B}}(x)\,{}_x\vec{\omega}(y) & \text{if } x \in \operatorname{supp}(\vec{\omega}_{\mathcal{B}}) \\ 0 & \text{if } x \notin \operatorname{supp}(\vec{\omega}_{\mathcal{B}}) \end{cases}$$

Of course, a similar formula holds for $\overleftarrow{\omega}(xy)$. These formulas can be used in reverse to obtain arbitrary probability weights on the forward and backward operational products. The technique is shown in the following theorem, the proof of which is straightforward.

Theorem 2.15. Let $\omega \in \Omega(\mathcal{A})$ and, for every $x \in \operatorname{supp}(\omega)$, let ${}_x\omega \in \Omega(\mathcal{B})$. Define $\vec{\omega}$: $XY \to [0, 1] \subseteq \mathbb{R}$ for $xy \in XY$ by

$$\vec{\omega}(xy) = \begin{cases} \omega(x)\,{}_x\omega(y) & \text{if } \ x \in \operatorname{supp}(\omega) \\ 0 & \text{if } \ x \notin \operatorname{supp}(\omega) \end{cases}$$

Then $\vec{\omega} \in \Omega(\overrightarrow{\mathcal{A}\mathcal{B}})$. Likewise, let $\eta \in \Omega(\mathcal{B})$ and, for every $y \in \operatorname{supp}(\eta)$, let $\eta_y \in \Omega(\mathcal{A})$. Define $\overleftarrow{\eta}$: $XY \to [0, 1] \subseteq \mathbb{R}$ for $xy \in XY$ by

$$\overleftarrow{\eta}(xy) = \begin{cases} \eta(y)\,\eta_y(x) & \text{if } \ y \in \operatorname{supp}(\eta) \\ 0 & \text{if } \ y \notin \operatorname{supp}(\eta) \end{cases}$$

Then $\overleftarrow{\eta} \in \Omega(\overleftarrow{\mathcal{A}\mathcal{B}})$.

In Theorem 2.15, it is clear that $\omega = \vec{\omega}_{\mathcal{B}}$, ${}_x\omega = {}_x\vec{\omega}$, $\eta = {}_{\mathcal{A}}\overleftarrow{\eta}$, and $\eta_y = \overleftarrow{\eta}_y$.

Notation 2.16. If $S = \text{supp}(\omega)$ and $T = \text{supp}(\eta)$, we denote the probability weights $\vec{\omega}$ and $\overleftarrow{\eta}$ in Theorem 2.15 by

$$\vec{\omega} = (\omega, ({}_x\omega)_{x\in S}), \qquad \overleftarrow{\eta} = (\eta, (\eta_y)_{y\in T})$$

Theorems 2.14 and 2.15 and the notation of 2.16 provide us with an effective computational tool for dealing with the probability weights on the forward and backward operational products.

Definition 2.17. We define the *pretensor product* of the quasimanuals $\mathscr{A}$ and $\mathscr{B}$, in symbols $\mathscr{A}\mathscr{B}$, by

$$\mathscr{A}\mathscr{B} = \overrightarrow{\mathscr{A}\mathscr{B}} \cup \overleftarrow{\mathscr{A}\mathscr{B}}$$

Foulis and Randall (1980) defined the *tensor product* $\mathscr{A} \otimes \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ to be a "manual closure" of the pretensor product (assuming that $\mathscr{A}$ and $\mathscr{B}$ satisfy certain mild regularity conditions). However, if $\mathscr{A} \otimes \mathscr{B}$ exists, then $\Omega(\mathscr{A}\mathscr{B}) = \Omega(\mathscr{A} \otimes \mathscr{B})$, and, since our concern in this paper is with probability weights and their linear span, we have no need for $\mathscr{A} \otimes \mathscr{B}$ here. Obviously,

$$\Omega(\mathscr{A}\mathscr{B}) = \Omega(\overrightarrow{\mathscr{A}\mathscr{B}}) \cap \Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$$

and so we have the following two theorems:

Theorem 2.18. Let $\omega: XY \to \mathbb{R}$. Then $\omega \in \Omega(\mathscr{A}\mathscr{B})$ if and only if $\omega \in \Omega(\mathscr{A} \times \mathscr{B})$, ω exhibits no influence of $\mathscr{A}$ on $\mathscr{B}$, and ω exhibits no influence of $\mathscr{B}$ on $\mathscr{A}$.

Proof. Theorems 2.10 and 2.12.

Theorem 2.19. Let $\omega: XY \to \mathbb{R}$. Then $\omega \in \Omega(\mathscr{A}\mathscr{B})$ if and only if there exist $\omega_{\mathscr{B}} \in \Omega(\mathscr{A})$ and ${}_{\mathscr{A}}\omega \in \Omega(\mathscr{B})$, and there exist ${}_x\omega \in \Omega(\mathscr{B})$ for every $x \in S = \text{supp}(\omega_{\mathscr{B}})$ and $\omega_y \in \Omega(\mathscr{A})$ for every $y \in T = \text{supp}({}_{\mathscr{A}}\omega)$ such that

$$\omega = (\omega_{\mathscr{B}}, ({}_x\omega)_{x\in S}) = ({}_{\mathscr{A}}\omega, (\omega_y)_{y\in T})$$

Proof. Lemma 2.14, Theorem 2.15, and the notation in 2.16.

Note that the condition in Theorem 2.19 implies that $\omega_{\mathscr{B}}$ and ${}_{\mathscr{A}}\omega$ are the respective reduced weights (marginals) for ω on $\mathscr{A}$ and $\mathscr{B}$, respectively. Furthermore, this condition requires that, for $x \in S$ and $y \in T$,

$$\omega_{\mathscr{B}}(x)\,{}_x\omega(y) = {}_{\mathscr{A}}\omega(y)\,\omega_y(x)$$

3. PRODUCT WEIGHTS

In this section we introduce *product weights*, and use them to prove the main theorem of the paper (Theorem 3.9). *We maintain our convention that $\mathscr{A}$ and $\mathscr{B}$ are quasimanuals with $X=\bigcup\mathscr{A}$ and $Y=\bigcup\mathscr{B}$.*

Definition 3.1. Let $\lambda\in V(\mathscr{A})$ and $\mu\in V(\mathscr{B})$. We define the *product* $\lambda\mu\colon XY\to\mathbb{R}$ by $(\lambda\mu)(xy)=\lambda(x)\mu(y)$.

If $\lambda\in\Omega(\mathscr{A})$ and $\mu\in\Omega(\mathscr{B})$, then it is clear that $\lambda\mu\in\Omega(\mathscr{A}\mathscr{B})$. Indeed, we have, with the notation of Theorem 2.19, $\lambda\mu=\omega$ with $\omega_{\mathscr{B}}=\lambda$, ${}_{\mathscr{A}}\omega=\mu$, ${}_x\omega=\mu$ for all $x\in S$, and $\omega_y=\lambda$ for all $y\in T$. Therefore, for $\lambda\in V(\mathscr{A})$ and $\mu\in V(\mathscr{B})$, it follows from the bilinearity of the mapping $(\lambda,\mu)\to\lambda\mu$ that $\lambda\mu\in V(\mathscr{A}\mathscr{B})$.

Definition 3.2. Let $V(\mathscr{A})\otimes V(\mathscr{B})$ denote the algebraic tensor product of the vector spaces $V(\mathscr{A})$ and $V(\mathscr{B})$. Then the unique linear map $V(\mathscr{A})\otimes V(\mathscr{B})\to V(\mathscr{A}\mathscr{B})$ such that $\lambda\otimes\mu\mapsto\lambda\mu$ for all $\lambda\in V(\mathscr{A})$ and all $\mu\in V(\mathscr{B})$ will be denoted by $\beta\colon V(\mathscr{A})\otimes V(\mathscr{B})\to V(\mathscr{A}\mathscr{B})$.

Lemma 3.3. $\beta\colon V(\mathscr{A})\otimes V(\mathscr{B})\to V(\mathscr{A}\mathscr{B})$ is a linear injection.

Proof. Let $\tau\in V(\mathscr{A})\otimes V(\mathscr{B})$ with $\beta(\tau)=0$. Select Hamel bases $(\lambda_i)_{i\in I}$ and $(\mu_j)_{j\in J}$ for $V(\mathscr{A})$ and $V(\mathscr{B})$, respectively. Then $\tau=\sum_{ij}t_{ij}(\lambda_i\otimes\mu_j)$, where the coefficients $t_{ij}\in\mathbb{R}$ are finitely nonzero. Since $\beta(\tau)=0$, $\sum t_{ij}\lambda_i(x)\mu_j(y)=0$ holds for all $x\in X$ and all $y\in Y$. Hence, for each $x\in X$, we have $\sum_j[\sum_i t_{ij}\lambda_i(x)]\mu_j=0$. Because the vectors $(\mu_j)_{j\in J}$ are linearly independent in $V(\mathscr{B})$, it follows that, for all $x\in X$ and all $j\in J$, $\sum_i t_{ij}\lambda_i(x)=0$. Therefore, for all $j\in J$, $\sum_i t_{ij}\lambda_i=0$. Because the vectors $(\lambda_i)_{i\in I}$ are linearly independent in $V(\mathscr{A})$, it follows that $t_{ij}=0$ for all $i\in I$ and all $j\in J$; hence, $\tau=0$.

The following lemma introduces a simple technical device which often proves useful when dealing with quasimanuals. We omit the proof, which is a routine verification.

Lemma 3.4. Let p be any object that does not belong to $X=\bigcup\mathscr{A}$, and let $\mathscr{A}^{\#}=\mathscr{A}\cup\{\{p\}\}$ be the quasimanual obtained by adjoining the singleton set $\{p\}$ to $\mathscr{A}$. Let $X^{\#}=\bigcup\mathscr{A}^{\#}=X\cup\{p\}$ and, for $\omega\in\Omega(\mathscr{A})$, define $\omega^{\#}\colon X^{\#}\to[0,1]\subseteq\mathbb{R}$ by $\omega^{\#}(x)=\omega(x)$ for $x\in X$ and $\omega^{\#}(p)=1$. Then $\omega\mapsto\omega^{\#}$ is an affine isomorphism of $\Omega(\mathscr{A})$ onto $\Omega(\mathscr{A}^{\#})$, and it extends uniquely to a linear isomorphism $V(\mathscr{A})\simeq V(\mathscr{A}^{\#})$.

The next lemma shows that the adjunction of a singleton set to each of the factors of a pretensor product, as in Lemma 3.4, has no effect on the algebraic structure of the space of probability weights on the pretensor product.

Lemma 3.5. Let p be an object that belongs neither to $X = \bigcup \mathscr{A}$ nor to $Y = \bigcup \mathscr{B}$ and let $\mathscr{A}^{\#} = \mathscr{A} \cup \{\{p\}\}$ and $\mathscr{B}^{\#} = \mathscr{B} \cup \{\{p\}\}$. Let $X^{\#} = \bigcup \mathscr{A}^{\#} = X \cup \{p\}$ and $Y^{\#} = \bigcup \mathscr{B}^{\#} = Y \cup \{p\}$. For $\omega \in \Omega(\mathscr{A}\mathscr{B})$, define ω': $X^{\#}Y^{\#} \to [0, 1] \subseteq \mathbb{R}$ by

$$\omega'(xy) = \begin{cases} \omega(xy) & \text{if} \quad x, y \neq p \\ \omega_{\mathscr{B}}(x) & \text{if} \quad x \neq p, y = p \\ {}_{\mathscr{A}}\omega(y) & \text{if} \quad x = p, y \neq p \\ 1 & \text{if} \quad x, y = p \end{cases}$$

Then $\omega \mapsto \omega'$ is an affine isomorphism of $\Omega(\mathscr{A}\mathscr{B})$ onto $\Omega(\mathscr{A}^{\#}\mathscr{B}^{\#})$, and it extends uniquely to a linear isomorphism $V(\mathscr{A}\mathscr{B}) \simeq V(\mathscr{A}^{\#}\mathscr{B}^{\#})$.

Proof. By a straightforward computation, $\omega' \in \Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$; hence, by symmetry, $\omega' \in \Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$, and it follows that $\omega' \in \Omega(\mathscr{A}\mathscr{B})$. Evidently, the restriction mapping $\Omega(\mathscr{A}^{\#}\mathscr{B}^{\#}) \to \Omega(\mathscr{A}\mathscr{B})$ is affine, and it is effective as the inverse of $\omega \mapsto \omega'$.

The following lemma is the key to the proof of the main theorem.

Lemma 3.6. Let Z be an $\mathscr{A}$-basic subset of X and let W be a $\mathscr{B}$-basic subset of Y. Then, with the notation of Lemma 3.5, $ZW \cup Zp \cup pW$ is an $\mathscr{A}^{\#}\mathscr{B}^{\#}$-basic subset of $X^{\#}Y^{\#}$.

Proof. Let $U = ZW \cup Zp \cup pW$. Note that Z is an $\mathscr{A}^{\#}$-basic subset of $X^{\#}$ and W is a $\mathscr{B}^{\#}$-basic subset of $Y^{\#}$. Suppose that $\alpha, \beta \in \Omega(\mathscr{A}^{\#}\mathscr{B}^{\#})$ and that $\alpha|_U = \beta|_U$. By Lemma 2.5, it will suffice to prove that $\alpha = \beta$. Define μ, μ': $X^{\#} \to [0, 1] \subseteq \mathbb{R}$ by $\mu(x) = \alpha(xp)$ and $\mu'(x) = \beta(xp)$ for all $x \in X^{\#}$. Since $\{p\}$ belongs to $\mathscr{B}^{\#}$, μ and μ' are just the reduced probability weights in $\Omega(\mathscr{A}^{\#})$ for α and β, respectively. Because $\alpha|_U = \beta|_U$, it follows that $\mu(z) = \mu'(z)$ for all $z \in Z$; hence, since Z is $\mathscr{A}^{\#}$-basic, we have $\mu = \mu'$. Consequently,

$$\exists \mu \in \Omega(\mathscr{A}^{\#}), \qquad \mu(x) = \alpha(xp) = \beta(xp), \qquad \forall x \in X^{\#} \tag{1}$$

By symmetry,

$$\exists \nu \in \Omega(\mathscr{B}^{\#}), \qquad \nu(y) = \alpha(py) = \beta(py), \qquad \forall y \in Y^{\#} \tag{2}$$

Now, let $S = \text{supp}(\mu) \subseteq X^{\#}$ and $T = \text{supp}(\nu) \subseteq Y^{\#}$. For $x \in S$, let

$${}_x\alpha(y) = \alpha(xy)/\mu(x), \qquad {}_x\beta(y) = \beta(xy)/\mu(y), \qquad \forall y \in Y^{\#}$$

noting that ${}_x\alpha$ is α preconditioned by x and ${}_x\beta$ is β preconditioned by x; hence, ${}_x\alpha$, ${}_x\beta \in \Omega(\mathscr{B}^{\#})$. Likewise, for $y \in T$, define

$$\alpha_y(x) = \alpha(xy)/\nu(y), \qquad \beta_y(x) = \beta(xy)/\nu(y), \qquad \forall x \in X^{\#}$$

noting that α_y is α postconditioned by y and β_y is β postconditioned by y; hence, $\alpha_y, \beta_y \in \Omega(\mathscr{A}^{\#})$. Therefore, for all $xy \in X^{\#}Y^{\#}$, we have

$$\alpha(xy) = \begin{cases} \mu(x)_x\alpha(y) & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S \end{cases} = \begin{cases} \alpha_y(x)\nu(y) & \text{if} \quad y \in T \\ 0 & \text{if} \quad y \notin T \end{cases} \tag{3}$$

$$\beta(xy) = \begin{cases} \mu(x)_x\beta(y) & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \notin S \end{cases} = \begin{cases} \beta_y(x)\nu(y) & \text{if} \quad y \in T \\ 0 & \text{if} \quad y \notin T \end{cases} \tag{4}$$

Claim:

$$w \in W \cap T \Rightarrow \alpha_w = \beta_w \tag{5}$$

To prove (5), note that, by (3) and (4), we have, for all $z \in Z$,

$$\alpha_w(z)\nu(w) = \alpha(zw) = \alpha|_U(zw) = \beta|_U(zw) = \beta(zw) = \beta_w(z)\nu(w)$$

Since $w \in T = \text{supp}(\nu)$, $\nu(w) \neq 0$, and it follows that

$$\alpha_w(z) = \beta_w(z), \qquad \forall z \in Z$$

The fact that Z is $\mathscr{A}^{\#}$-basic implies (5).

Claim:

$$w \in W \cap T, \quad x \in S \Rightarrow {}_x\alpha(w) = {}_x\beta(w) \tag{6}$$

To prove (6), note that, by (3), (5), and (4), we have

$$\mu(x)_x\alpha(w) = \alpha_w(x)\nu(w) = \beta_w(x)\nu(w) = \mu(x)_x\beta(w)$$

Since $x \in S = \text{supp}(\mu)$, $\mu(x) \neq 0$, and (6) follows.

Claim:

$$w \in W \backslash T, \quad x \in S \Rightarrow {}_x\alpha(w) = {}_x\beta(w) = 0 \tag{7}$$

To prove (7), note that by (3) and the supposition that $w \notin T$, $\mu(x)_x\alpha(w) = 0$; hence, because $x \in S$ so that $\mu(x) \neq 0$, we have ${}_x\alpha(w) = 0$. Likewise, by (4) and the supposition that $w \notin T$, $\mu(x)_x\beta(w) = 0$; hence, ${}_x\beta(w) = 0$.

Combining (6) and (7), we have

$$w \in W, \quad x \in S \Rightarrow {}_x\alpha(w) = {}_x\beta(w)$$

Because W is $\mathscr{B}^{\#}$-basic, it follows that ${}_x\alpha = {}_x\beta$ holds for all $x \in S$. Therefore, by (3) and (4), $\alpha = \beta$, and the proof is complete.

Definition 3.7. If $V(\mathscr{A})$ is finite-dimensional, we say that $\mathscr{A}$ is a *finite-dimensional* quasimanual and we define

$$\dim(\mathscr{A}) = \dim V(\mathscr{A})$$

Theorem 3.8. If $\mathscr{A}$ and $\mathscr{B}$ are finite-dimensional, then

$$\dim(\mathscr{A}\mathscr{B}) = \dim(\mathscr{A}) \cdot \dim(\mathscr{B})$$

Proof. By Lemma 3.3 and the fact that the dimension of the algebraic tensor product $V(\mathscr{A})\otimes V(\mathscr{B})$ is the product of the dimensions of the factors, we have

$$\dim(\mathscr{A})\cdot\dim(\mathscr{B})\leq\dim(\mathscr{A}\mathscr{B})$$

Select finite $Z\subseteq X$ and $W\subseteq Y$ so that

$$\{f_z|z\in Z\}\cup\{e_{\mathscr{A}}\}\quad\text{is a basis for } V(\mathscr{A})^*$$

and

$$\{f_w|w\in W\}\cup\{e_{\mathscr{B}}\}\quad\text{is a basis for } V(\mathscr{B})^*$$

where $e_{\mathscr{A}}$ and $e_{\mathscr{B}}$ are the dual order units for $V(\mathscr{A})$ and $V(\mathscr{B})$, respectively. Let $m=\#Z$ and $n=\#W$. Then,

$$\dim(\mathscr{A})=\#Z+1=m+1,\qquad \dim(\mathscr{B})=\#W+1=n+1$$

With the notation of Lemma 3.6, it follows that $U=ZW\cup Zp\cup pW$ is $\mathscr{A}^{\#}\mathscr{B}^{\#}$-basic. Therefore, by Lemmas 3.5 and 3.6,

$$\begin{aligned}\dim(\mathscr{A}\mathscr{B})&=\dim(\mathscr{A}^{\#}\mathscr{B}^{\#})\leq\#U+1=mn+m+n+1=(m+1)(n+1)\\&=\dim(\mathscr{A})\cdot\dim(\mathscr{B})\end{aligned}$$

We are now in a position to state the main theorem of the paper.

Theorem 3.9. If $\mathscr{A}$ and $\mathscr{B}$ are finite-dimensional, then

$$\beta\colon V(\mathscr{A})\otimes V(\mathscr{B})\to V(\mathscr{A}\mathscr{B})$$

is a vector space isomorphism.

Proof. Lemma 3.3 and Theorem 3.8.

Corollary 3.10. If $\mathscr{A}$ and $\mathscr{B}$ are finite-dimensional, then $\Omega(\mathscr{A}\mathscr{B})$ consists of all finite affine combinations

$$\omega=\sum t_i\lambda_i\mu_i,\qquad t_i\in\mathbb{R},\quad \sum t_i=1$$

of products $\lambda_i\mu_i$ of probability weights $\lambda_i\in\Omega(\mathscr{A})$, $\mu_i\in\Omega(\mathscr{B})$ such that

$$\omega(xy)\geq 0\qquad\text{for all}\quad x\in X\quad\text{and all}\quad y\in Y$$

Proof. Because $V(\mathscr{A})$ is the linear span of $\Omega(\mathscr{A})$ and $V(\mathscr{B})$ is the linear span of $\Omega(\mathscr{B})$, it follows that $V(\mathscr{A})\otimes V(\mathscr{B})$ is the linear span of all pure tensors of the form $\lambda\otimes\mu$ with $\lambda\in\Omega(\mathscr{A})$ and $\mu\in\Omega(\mathscr{B})$. Consequently, by Theorem 3.9, $V(\mathscr{A}\mathscr{B})$ is the linear span of all product probability weights of the form $\lambda\mu$ with $\lambda\in\Omega(\mathscr{A})$, $\mu\in\Omega(\mathscr{B})$. Thus, $\omega\in\Omega(\mathscr{A}\mathscr{B})$ can be written in the form $\omega=\sum t_i\lambda_i\mu_i$ with $\lambda_i\in\Omega(\mathscr{A})$, $\mu_i\in\Omega(\mathscr{B})$. Applying the dual order unit $e \in V(\mathscr{A}\mathscr{B})^*$ to both sides of the equation $\omega = \sum t_i \lambda_i \mu_i$ and noting that $e(\lambda_i \mu_i) = 1$, we find that $\sum t_i = 1$. Since ω is a probability weight, $\omega(xy) \geq 0$ holds for all $x \in X$, $y \in Y$. Conversely, the fact that any ω of the indicated form is a probability weight on $\mathscr{A}\mathscr{B}$ if $\omega(xy) \geq 0$ for all $xy \in XY$ is obvious.

4. EXAMPLES—UNITARY AND EUCLIDEAN SPACES

In this section, we consider some of the consequences of the theory developed above for the probability measures associated with finite-dimensional Hilbert spaces over $\mathbb{C}$ (unitary spaces) or over $\mathbb{R}$ (Euclidean spaces). We consider only spaces of dimension three or more, so that Gleason's theorem applies. If $\mathscr{H}$ is such a Hilbert space, then $\mathscr{L}(\mathscr{H})$ denotes the vector space of all linear operators on $\mathscr{H}$ and $\mathscr{V}(\mathscr{H})$ denotes the real vector space of all self-adjoint operators on $\mathscr{H}$.

If $\mathscr{H}$ is Euclidean, we define $\mathscr{V}_\perp(\mathscr{H})$ to be the real vector space of all skew-adjoint operators on $\mathscr{H}$. Note that $\mathscr{V}_\perp(\mathscr{H})$ is the orthogonal complement of $\mathscr{V}(\mathscr{H})$ with respect to the inner product

$$\langle A, B \rangle = \text{trace}(AB^*) \qquad \text{for} \quad A, B \in \mathscr{L}(\mathscr{H})$$

so that, in the Euclidean case, we have $\mathscr{L}(\mathscr{H}) = \mathscr{V}(\mathscr{H}) \oplus \mathscr{V}_\perp(\mathscr{H})$.

For simplicity, we call an operator $A \in \mathscr{V}(\mathscr{H})$ *positive* if it is positive-semidefinite in the sense that $\langle A\psi, \psi \rangle \geq 0$ holds for all vectors $\psi \in \mathscr{H}$. A *density operator* on $\mathscr{H}$ is a positive operator of trace one in $\mathscr{V}(\mathscr{H})$, and the convex subset of $\mathscr{V}(\mathscr{H})$ consisting of all such density operators is denoted by $\mathscr{W}(\mathscr{H})$. Thus, with $\mathscr{W}(\mathscr{H})$ as a cone base, $\mathscr{V}(\mathscr{H})$ forms a base-norm space.

The set $\mathscr{A}$ of all maximal orthonormal subsets of $\mathscr{H}$ is called the *frame manual* of $\mathscr{H}$. As mentioned earlier, there is a natural linear isomorphism $A \mapsto \omega_A$ from $\mathscr{V}(\mathscr{H})$ onto $V(\mathscr{A})$ given by

$$\omega_A(\psi) = \text{trace}(AP_\psi) = \langle A\psi, \psi \rangle$$

for all unit vectors $\psi \in \mathscr{H}$, and this isomorphism carries $\mathscr{W}(\mathscr{H})$ onto the cone base $\Omega(\mathscr{A})$ of probability weights in $V(\mathscr{A})$.

Now let $\mathscr{H}_1$ and $\mathscr{H}_2$ be finite-dimensional Hilbert spaces, either both unitary or both Euclidean. For $i = 1, 2$ we denote the frame manual of $\mathscr{H}_i$ by $\mathscr{A}_i$. Let $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ be the Hilbert-space tensor product and let $\mathscr{A}$ be the frame manual of $\mathscr{H}$. There is a natural linear isomorphism

$$\mathscr{L}(\mathscr{H}_1) \otimes \mathscr{L}(\mathscr{H}_2) \simeq \mathscr{L}(\mathscr{H}_1 \otimes \mathscr{H}_2)$$

such that the algebraic tensor product $A_1 \otimes A_2$ of the operators $A_1 \in \mathscr{L}(\mathscr{H}_1)$ and $A_2 \in \mathscr{L}(\mathscr{H}_2)$ corresponds to the Kronecker product of A_1 and A_2. In what follows, we simply identify $A_1 \otimes A_2$ with this Kronecker product, so that, for $\phi \in \mathscr{H}_1$, $\psi \in \mathscr{H}_2$,

$$(A_1 \otimes A_2)(\phi \otimes \psi) = A_1(\phi) \otimes A_2(\psi)$$

In particular, we shall regard $\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)$ as a linear subspace of $\mathscr{V}(\mathscr{H})=\mathscr{V}(\mathscr{H}_1\otimes\mathscr{H}_2)$.

Note that, if $E\in\mathscr{A}_1\mathscr{A}_2$, then $\{\phi\otimes\psi|\phi\psi\in E\}\in\mathscr{A}$. Therefore, the mapping ξ: $V(\mathscr{A})\to V(\mathscr{A}_1\mathscr{A}_2)$ defined by $\xi(\nu)(\phi\psi)=\nu(\phi\otimes\psi)$ for $\nu\in V(\mathscr{A})$, $\phi\psi\in\bigcup(\mathscr{A}_1\mathscr{A}_2)$, is linear and maps the cone base $\Omega(\mathscr{A})$ in $V(\mathscr{A})$ into the cone base $\Omega(\mathscr{A}_1\mathscr{A}_2)$ in $V(\mathscr{A}_1\mathscr{A}_2)$. Hence, by composing the natural linear isomorphism $A\mapsto\omega_A$ from $\mathscr{V}(\mathscr{H})$ onto $V(\mathscr{A})$ with ξ, we obtain a linear mapping τ: $\mathscr{V}(\mathscr{H})\to V(\mathscr{A}_1\mathscr{A}_2)$, denoted by $A\mapsto\tau_A$, such that

$$\tau_A(\phi\psi)=\xi(\omega_A)(\phi\psi)=\omega_A(\phi\otimes\psi)=\langle A(\phi\otimes\psi),\phi\otimes\psi\rangle$$

holds for all $A\in\mathscr{V}(\mathscr{H})$ and all unit vectors $\phi\in\mathscr{H}_1$, $\psi\in\mathscr{H}_2$. Evidently, τ maps the cone base $\mathscr{W}(\mathscr{H})$ of $\mathscr{V}(\mathscr{H})$ into the cone base $\Omega(\mathscr{A}_1\mathscr{A}_2)$ of $V(\mathscr{A}_1\mathscr{A}_2)$.

Because $\mathscr{V}(\mathscr{H}_i)\simeq V(\mathscr{A}_i)$ for $i=1,2$, it follows from Theorem 3.9 that

$$\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)\simeq V(\mathscr{A}_1)\otimes V(\mathscr{A}_2)\simeq V(\mathscr{A}_1\mathscr{A}_2)$$

In fact, the restriction of τ: $\mathscr{V}(\mathscr{H})\to V(\mathscr{A}_1\mathscr{A}_2)$ to the linear subspace $\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)$ of $\mathscr{V}(\mathscr{H})$ provides just such a linear isomorphism $A\mapsto\tau_A$ for $A\in\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)$. In particular, if $A_1\in\mathscr{V}(\mathscr{H}_1)$, $A_2\in\mathscr{V}(\mathscr{H}_2)$, $A=A_1\otimes A_2$, $\phi\in\bigcup\mathscr{A}_1$, and $\psi\in\bigcup\mathscr{A}_2$, then we have

$$\tau_A(\phi\psi)=\langle A_1\phi,\phi\rangle\langle A_2\psi,\psi\rangle$$

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are unitary spaces, then $\mathscr{V}(\mathscr{H})=\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)$; hence, *in the unitary case*, $A\mapsto\tau_A$ provides a linear isomorphism

$$\mathscr{V}(\mathscr{H})\simeq V(\mathscr{A}_1\mathscr{A}_2)$$

However, *in the Euclidean case,*

$$\mathscr{V}(\mathscr{H})=[\mathscr{V}(\mathscr{H}_1)\otimes\mathscr{V}(\mathscr{H}_2)]\oplus[\mathscr{V}_\perp(\mathscr{H}_1)\otimes\mathscr{V}_\perp(\mathscr{H}_2)]$$

and, in this case, $\mathscr{V}_\perp(\mathscr{H}_1)\otimes\mathscr{V}_\perp(\mathscr{H}_2)$ is precisely the null space of the linear mapping τ: $\mathscr{V}(\mathscr{H})\to V(\mathscr{A}_1\mathscr{A}_2)$. Thus, in the Euclidean case, $V(\mathscr{A}_1\mathscr{A}_2)$ is isomorphic to a *proper* subspace of $\mathscr{V}(\mathscr{H})$.

Now, we investigate the question of how the various cone bases fare under the linear mappings introduced above. Say that an operator $A\in\mathscr{V}(\mathscr{H})$ is *positive on pure tensors* (PPT) if

$$\langle A(\psi\otimes\phi),\psi\otimes\phi\rangle\geq 0$$

holds for all $\psi\in\mathscr{H}_1$ and all $\phi\in\mathscr{H}_2$. Obviously, every positive operator in $\mathscr{V}(\mathscr{H})$ is PPT; however, there are simple examples of operators in $\mathscr{V}(\mathscr{H})$ that are PPT, but not positive. Define the convex subset $\mathscr{C}(\mathscr{H})$ of $\mathscr{V}(\mathscr{H})$ by

$$\mathscr{C}(\mathscr{H})=\{C\in\mathscr{V}(\mathscr{H})|C\text{ is PPT and trace}(C)=1\}$$

Note that $\mathscr{W}(\mathscr{H})$ is a proper subset of $\mathscr{C}(\mathscr{H})$. Evidently, τ maps $\mathscr{C}(\mathscr{H})$ onto the cone base $\Omega(\mathscr{A}_1\mathscr{A}_2)$. Define

$$\mathscr{T}(\mathscr{H}) = \mathscr{C}(\mathscr{H}) \cap [\mathscr{V}(\mathscr{H}_1) \otimes \mathscr{V}(\mathscr{H}_2)]$$

$\mathscr{T}(\mathscr{H})$ consists of all PPT operators in $\mathscr{V}(\mathscr{H})$ that are affine combinations of tensor products of density operators on $\mathscr{H}_1$ and $\mathscr{H}_2$. In conformity with Corollary 3.10, the restriction of τ to $\mathscr{T}(\mathscr{H})$ is an affine isomorphism of $\mathscr{T}(\mathscr{H})$ onto $\Omega(\mathscr{A}_1\mathscr{A}_2)$. Thus, we have the following theorem:

Theorem 4.1. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be unitary or Euclidean spaces of dimensions three or more, and let $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. Let $\mathscr{A}_1$, $\mathscr{A}_2$, and $\mathscr{A}$ be the frame manuals of $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}$, respectively. Let $\mathscr{C}(\mathscr{H})$ be the convex set of all self-adjoint operators on $\mathscr{H}$ that are positive on pure tensors and have unit trace. Let $\mathscr{T}(\mathscr{H})$ denote the convex set consisting of those operators in $\mathscr{C}(\mathscr{H})$ that are affine combinations of tensor products of density operators on $\mathscr{H}_1$ and $\mathscr{H}_2$. Let $\mathscr{W}(\mathscr{H})$ be the convex set of all density operators on $\mathscr{H}$. Then $\mathscr{W}(\mathscr{H}) \cup \mathscr{T}(\mathscr{H}) \subseteq \mathscr{C}(\mathscr{H})$, and there are affine isomorphisms

$$\mathscr{W}(\mathscr{H}) \simeq \Omega(\mathscr{A}) \qquad \text{and} \qquad \mathscr{T}(\mathscr{H}) \simeq \Omega(\mathscr{A}_1\mathscr{A}_2)$$

provided by the mappings $A \mapsto \omega_A$ and $A \mapsto \tau_A$, respectively.

In the unitary case, we have

$$\mathscr{W}(\mathscr{H}) \subset \mathscr{T}(\mathscr{H}) = \mathscr{C}(\mathscr{H})$$

and $\mathscr{T}(\mathscr{H}) \setminus \mathscr{W}(\mathscr{H})$ is nonempty. Thus, all of the probability measures affiliated with density operators on the tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ of unitary spaces are represented in the space $\Omega(\mathscr{A}_1\mathscr{A}_2)$ of probability weights on the pretensor product $\mathscr{A}_1\mathscr{A}_2$ of the corresponding frame manuals; however, there are probability weights in $\Omega(\mathscr{A}_1\mathscr{A}_2)$ that do not correspond to any of the probability measures affiliated with density operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$. The probability weights in $\mathscr{T}(\mathscr{H}) \setminus \mathscr{W}(\mathscr{H})$ correspond to the "anomalous states" that have caused concern because they seem to assign "negative probabilities" to certain propositions affiliated with quantum mechanical systems (Groenewold, 1985). The formalism set forth in this paper shows how such anomalous states can arise, how they can be interpreted, and how they can be dealt with mathematically.

Although the Euclidean case is perhaps not as significant as the unitary case from the point of view of orthodox quantum mechanics, it is even more interesting than the unitary case from a purely mathematical standpoint. In the Euclidean case, $\mathscr{C}(\mathscr{H})$ is a disjoint union

$$\mathscr{C}(\mathscr{H}) = \mathscr{T}(\mathscr{H}) \setminus \mathscr{W}(\mathscr{H}) \,\dot{\cup}\, \mathscr{T}(\mathscr{H}) \cap \mathscr{W}(\mathscr{H}) \,\dot{\cup}\, \mathscr{W}(\mathscr{H}) \setminus \mathscr{T}(\mathscr{H})$$
$$\dot{\cup}\, \mathscr{C}(\mathscr{H}) \setminus (\mathscr{T}(\mathscr{H}) \cup \mathscr{W}(\mathscr{H}))$$

of four nonempty subsets. Thus, for Euclidean spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, we not only have anomalous states corresponding to elements in $\mathscr{T}(\mathscr{H})\setminus\mathscr{W}(\mathscr{H})$, we also have, vice versa, probability measures represented by elements of $\mathscr{W}(\mathscr{H})\setminus\mathscr{T}(\mathscr{H})$ that have no counterparts in $\Omega(\mathscr{A}_1\mathscr{A}_2)$.

5. THE TOTALLY FINITE CASE

A quasimanual $\mathscr{A}$ is said to be *finite* if there are only finitely many sets $E\in\mathscr{A}$; *locally finite* if every $E\in\mathscr{A}$ is a finite set; and *totally finite* if it is both finite and locally finite. Thus, $\mathscr{A}$ is totally finite if and only if $\bigcup\mathscr{A}$ is a finite set.

Definition 5.1. The set of probability weights $\Omega(\mathscr{A})$ for the quasimanual $\mathscr{A}$ is said to be *positive* if, for each $x\in\bigcup\mathscr{A}$, there exists $\omega\in\Omega(\mathscr{A})$ such that $\omega(x)>0$. A probability weight $\omega\in\Omega(\mathscr{A})$ is said to be *strictly positive* if $\omega(x)>0$ holds for every $x\in\bigcup\mathscr{A}$.

Lemma 5.2. Let the totally finite quasimanual $\mathscr{A}$ have a positive set of probability weights $\Omega(\mathscr{A})$. Then there is a strictly positive probability weight $\alpha\in\Omega(\mathscr{A})$.

Proof. For each $x\in X=\bigcup\mathscr{A}$, select $\omega_x\in\Omega(\mathscr{A})$ such that $\omega_x(x)>0$. With $n=\#X$, put

$$\alpha=\left(\frac{1}{n}\right)\sum_{x\in X}\omega_x$$

Lemma 5.3. Let α be a strictly finite probability weight on the totally finite quasimanual $\mathscr{A}$. Let $X=\bigcup\mathscr{A}$, let $\lambda\in\mathbb{R}^X$, and suppose that there exists a fixed $c\in\mathbb{R}$ such that

$$\sum_{x\in E}\lambda(x)=c \qquad \text{for every} \quad E\in\mathscr{A}$$

Then:

(i) There exist $s, t\in\mathbb{R}$ with $s, t>0$ and there exists $\omega\in\Omega(\mathscr{A})$ such that $\lambda=s\alpha-t\omega$.

(ii) $\lambda\in V(\mathscr{A})$.

Proof. Because X is finite, there exists $s>0$ such that $\lambda(x)<s\alpha(x)$ holds for all $x\in X$. Let $\mu\in\mathbb{R}^X$ be defined by $\mu=s\alpha-\lambda$, and let $t=\sum_{x\in E}\mu(x)$ for any $E\in\mathscr{A}$. Let $\omega=(1/t)\mu$, noting that $\omega\in\Omega(\mathscr{A})$. This proves (i), and (ii) is an immediate consequence.

If $\mathscr{A}$ is a totally finite quasimanual with a positive set $\Omega(\mathscr{A})$ of probability weights, then $\Omega(\mathscr{A})$ is a polytope in $V(\mathscr{A})$ and the hyperplane $e^{-1}(1)$ is the affine span of $\Omega(\mathscr{A})$ (Gudder *et al.*, 1987); hence, the affine dimension of $\Omega(\mathscr{A})$ is $\dim(\mathscr{A})-1$. Conversely, by a theorem of Schultz (1984), every rational polytope is of the form $\Omega(\mathscr{A})$ for a suitable choice of the totally finite quasimanual $\mathscr{A}$.

Definition 5.4. Let $\mathscr{A}$ be a totally finite quasimanual with $X=\bigcup \mathscr{A}$. The linear transformation $T\colon \mathbb{R}^X \to \mathbb{R}^{\mathscr{A}}$ defined by

$$(T(\psi))(E)=\sum_{x\in E}\psi(x) \qquad \text{for} \quad \psi\in\mathbb{R}^X, \quad E\in\mathscr{A}$$

is called the *incidence transformation* for $\mathscr{A}$. The rank of T is called the *rank* of the quasimanual $\mathscr{A}$.

The matrix $(t_{x,E})$ for the incidence transformation T with respect to the Kronecker bases for $\mathbb{R}^X$ and $\mathbb{R}^{\mathscr{A}}$ is the *incidence matrix* for the quasimanual $\mathscr{A}$. Evidently,

$$t_{x,E}=\begin{cases}1 & \text{if} \quad x\in E\\ 0 & \text{if} \quad x\notin E\end{cases} \qquad \text{for} \quad x\in X, \quad E\in\mathscr{A}$$

The proof of the following theorem first appeared in Kläy (1985).

Theorem 5.5. Let $\mathscr{A}$ be a totally finite quasimanual with a positive set of probability weights and let $X=\bigcup\mathscr{A}$. Then

$$\dim(\mathscr{A})=1+\#X-\operatorname{rank}(\mathscr{A})$$

Proof. By part (ii) of Lemma 5.3, the null space of T coincides with the null space of e; hence,

$$\begin{aligned}\dim(\mathscr{A})&=\dim(V(\mathscr{A}))=1+\operatorname{nullity}(T)\\&=1+\dim(\mathbb{R}^X)-\operatorname{rank}(T)\\&=1+\#X-\operatorname{rank}(\mathscr{A})\end{aligned}$$

For the remainder of this section we assume that *$\mathscr{A}$ and $\mathscr{B}$ are totally finite quasimanuals with $X=\bigcup\mathscr{A}$ and $Y=\bigcup\mathscr{B}$*. We make the standing assumption that *both $\mathscr{A}$ and $\mathscr{B}$ carry positive sets of probability weights.* Therefore, by Lemma 5.2, there exist strictly positive probability weights $\alpha\in\Omega(\mathscr{A})$, $\beta\in\Omega(\mathscr{B})$; hence, the product weight $\alpha\beta$ is a strictly positive probability weight in $\Omega(\mathscr{A}\mathscr{B})$. Likewise, $\alpha\beta$ is a strictly positive probability weight in $\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$, $\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$, and $\Omega(\mathscr{A}\times\mathscr{B})$; hence, $\mathscr{A}\mathscr{B}$, $\overrightarrow{\mathscr{A}\mathscr{B}}$, $\overleftarrow{\mathscr{A}\mathscr{B}}$, and $\mathscr{A}\times\mathscr{B}$ are totally finite quasimanuals carrying positive sets of probability weights.

Theorem 5.6. $\operatorname{rank}(\mathscr{A}\times\mathscr{B})=\operatorname{rank}(\mathscr{A})\cdot\operatorname{rank}(\mathscr{B})$.

Proof. The incidence matrix of $\mathscr{A}\times\mathscr{B}$ is the Kronecker product of the incidence matrices of $\mathscr{A}$ and $\mathscr{B}$.

In the following theorem, we use the notations $e_{\mathscr{A}}$ and $e_{\mathscr{B}}$ for the dual order units $e_{\mathscr{A}} \in V(\mathscr{A})^*$ and $e_{\mathscr{B}} \in V(\mathscr{B})^*$.

Theorem 5.7. Let $U = \{\phi \in V(\mathscr{B})^X | e_{\mathscr{B}} \circ \phi \in V(\mathscr{A})\}$. Then U is a vector subspace of $V(\mathscr{B})^X$ and the mapping Ψ: $V(\overrightarrow{\mathscr{A}\mathscr{B}}) \to (\mathbb{R}^Y)^X$ defined by $((\Psi(\nu))(x))(y) = \nu(xy)$ for $\nu \in V(\overrightarrow{\mathscr{A}\mathscr{B}})$, $x \in X$, and $y \in Y$, is a linear isomorphism of $V(\overrightarrow{\mathscr{A}\mathscr{B}})$ onto U.

Proof. Let $\nu \in V(\overrightarrow{\mathscr{A}\mathscr{B}})$ and let $\phi = \Psi(\nu)$. We begin by showing that $\phi \in V(\mathscr{B})^X$. There exist $\vec{\omega}, \vec{\lambda} \in \Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ and nonnegative real numbers s and t such that $\nu = s\vec{\omega} - t\vec{\lambda}$. Also, with the notation of 2.16, $\vec{\omega} = (\omega_{\mathscr{B}}, ({}_x\omega)_{x \in S})$, $\vec{\lambda} = (\lambda_{\mathscr{B}}, ({}_x\lambda)_{x \in T})$, where $S = \operatorname{supp} \omega_{\mathscr{B}}$ and $T = \operatorname{supp} \lambda_{\mathscr{B}}$. Thus, for each $x \in X \cap T$,

$$\phi(x) = s\omega_{\mathscr{B}}(x)_x\omega - t\lambda_{\mathscr{B}}(x)_x\lambda \in V(\mathscr{B})$$

If $x \in S \backslash T$, then $\phi(x) = s\omega_{\mathscr{B}}(x)_x\omega \in V(\mathscr{B})$, if $x \in T \backslash S$, then $\phi(x) = -t\lambda_{\mathscr{B}}(x)_x\lambda \in V(\mathscr{B})$, and, if $x \in X \backslash (T \cup S)$, then $\phi(x) = 0 \in V(\mathscr{B})$. Therefore, $\phi \in V(\mathscr{B})^X$. Furthermore,

$$e_{\mathscr{B}}(\phi(x)) = s\omega_{\mathscr{B}}(x) - t\lambda_{\mathscr{B}}(x)$$

and it follows that $e_{\mathscr{B}} \circ \phi = s\omega_{\mathscr{B}} - t\lambda_{\mathscr{B}} \in V(\mathscr{A})$. Therefore, $\Psi(\nu) = \phi \in U$.

Evidently, Ψ: $V(\overrightarrow{\mathscr{A}\mathscr{B}}) \to U$ is linear and injective. We complete the proof by showing that it is surjective. Thus, suppose that $\phi \in U$ and define $\nu \in \mathbb{R}^{XY}$ by $\nu(xy) = (\phi(x))(y)$ for all $xy \in XY$. It is sufficient to prove that $\nu \in V(\overrightarrow{\mathscr{A}\mathscr{B}})$, for then, $\Psi(\nu) = \phi$. Let $G \in \overrightarrow{\mathscr{A}\mathscr{B}}$, so that there exists $E \in \mathscr{A}$ and, for each $x \in E$, there exists $F_x \in \mathscr{B}$ such that $G = \bigcup_{x \in E} xF_x$. Let

$$c = \sum_{xy \in G} \nu(xy) = \sum_{xy \in G} (\phi(x))(y)$$

By Lemma 5.3, it is enough to show that c is independent of the choice of $G \in \overrightarrow{\mathscr{A}\mathscr{B}}$. Because $\phi \in U$, we have $\phi(x) \in V(\mathscr{B})$ for every $x \in X$; hence,

$$c = \sum_{xy \in G} (\phi(x))(y) = \sum_{x \in E} \left[\sum_{y \in F_x} (\phi(x))(y) \right] = \sum_{x \in E} e_{\mathscr{B}}(\phi(x))$$
$$= \sum_{x \in E} (e_{\mathscr{B}} \circ \phi)(x) = e_{\mathscr{A}}(e_{\mathscr{B}} \circ \phi)$$

where, in the last step of the computation, we have used the fact that $e_{\mathscr{B}} \circ \phi \in V(\mathscr{A})$.

Theorem 5.8. $\dim(\overrightarrow{\mathscr{A}\mathscr{B}}) = \dim(\mathscr{A}) + [\dim(\mathscr{B}) - 1](\# X)$.

Proof. Let U be defined as in Theorem 5.7, noting that $\dim(U) = \dim(\overrightarrow{\mathscr{A}\mathscr{B}})$. Define L: $V(\mathscr{B})^X \to \mathbb{R}^X$ by $L(\phi) = e_{\mathscr{B}} \circ \phi$ for $\phi \in V(\mathscr{B})^X$. Evidently, L is linear and $U = L^{-1}(V(\mathscr{A}))$. We claim that L: $V(\mathscr{B})^X \to \mathbb{R}^X$

is surjective. Indeed, let $\psi \in \mathbb{R}^X$ and, for each $x \in X$, select $\beta_x \in V(\mathscr{B})$ such that $e_{\mathscr{B}}(\beta_x) = \psi(x)$. [This is possible because the nonzero linear functional $e_{\mathscr{B}}$: $V(\mathscr{B}) \to \mathbb{R}$ is surjective.] Define $\phi \in V(\mathscr{B})^X$ by $\phi(x) = \beta_x$ for each $x \in X$. Then, for $x \in X$,

$$(L(\phi))(x) = (e_{\mathscr{B}} \circ \phi)(x) = e_{\mathscr{B}}(\phi(x)) = e_{\mathscr{B}}(\beta_x) = \psi(x)$$

Hence, $L(\phi) = \psi$. Let $\Lambda \subseteq V(\mathscr{B})^X$ be the null space of L and let $\Delta \subseteq V(\mathscr{B})$ be the null space of $e_{\mathscr{B}}$. Evidently, $\Lambda = \Delta^X$; hence,

$$\dim(\Lambda) = [\dim(V(\mathscr{B})) - 1](\# X) = [\dim(\mathscr{B}) - 1](\# X)$$

Because Λ is a linear subspace of $U = L^{-1}(V(\mathscr{A}))$, we have $\dim(U) = \dim(U/\Lambda) + \dim(\Lambda)$; hence, it suffices to prove that $\dim(U/\Lambda) = \dim(\mathscr{A})$. However, because L: $V(\mathscr{B})^X \to \mathbb{R}^X$ is surjective and $V(\mathscr{A})$ is a linear subspace of $\mathbb{R}^X$,

$$U/\Lambda = L^{-1}(V(\mathscr{A}))/L^{-1}(0) \simeq V(\mathscr{A})$$

and, therefore, $\dim(U/\Lambda) = \dim(V(\mathscr{A})) = \dim(\mathscr{A})$.

Corollary 5.9. $\dim(\overleftarrow{\mathscr{A}\mathscr{B}}) = \dim(\mathscr{B}) + [\dim(\mathscr{A}) - 1](\# Y)$.

Theorem 5.10. $\dim(\mathscr{A} \times \mathscr{B}) + \dim(\mathscr{A}\mathscr{B}) = \dim(\overrightarrow{\mathscr{A}\mathscr{B}}) + \dim(\overleftarrow{\mathscr{A}\mathscr{B}})$.

Proof. Let $a = \dim(\mathscr{A})$, $b = \dim(\mathscr{B})$, $r_a = \operatorname{rank}(\mathscr{A})$, $r_b = \operatorname{rank}(\mathscr{B})$, $x = \# X$, and $y = \# Y$. By Theorem 5.5,

$$r_a = 1 + x - a \qquad \text{and} \qquad r_b = 1 + y - b$$

Thus, by Theorems 5.5 and 5.6,

$$\begin{aligned} \dim(\mathscr{A} \times \mathscr{B}) &= 1 + \#(XY) - \operatorname{rank}(\mathscr{A} \times \mathscr{B}) = 1 + xy - r_a r_b \\ &= 1 + xy - (1 + x - a)(1 + y - b) \\ &= a + (b-1)x + b + (a-1)y - ab \end{aligned}$$

Therefore, by Theorem 3.8, Theorem 5.8, and Corollary 5.9,

$$\dim(\mathscr{A} \times \mathscr{B}) + \dim(\mathscr{A}\mathscr{B}) = \dim(\overrightarrow{\mathscr{A}\mathscr{B}}) + \dim(\overleftarrow{\mathscr{A}\mathscr{B}})$$

Lemma 5.11. $V(\mathscr{A}\mathscr{B}) = V(\overrightarrow{\mathscr{A}\mathscr{B}}) \cap V(\overleftarrow{\mathscr{A}\mathscr{B}})$.

Proof. This result is an immediate consequence of Lemma 5.3, Definition 2.17, and the fact that $\overrightarrow{\mathscr{A}\mathscr{B}} \cap \overleftarrow{\mathscr{A}\mathscr{B}}$ is nonempty.

Because $\mathscr{A} \times \mathscr{B} \subseteq \overrightarrow{\mathscr{A}\mathscr{B}}$, it follows that $\Omega(\overrightarrow{\mathscr{A}\mathscr{B}}) \subseteq \Omega(\mathscr{A} \times \mathscr{B})$; hence, that $V(\overrightarrow{\mathscr{A}\mathscr{B}}) \subseteq V(\mathscr{A} \times \mathscr{B})$. Likewise, $V(\overleftarrow{\mathscr{A}\mathscr{B}}) \subseteq V(\mathscr{A} \times \mathscr{B})$, and it follows that, as subspaces of $\mathbb{R}^{XY}$, $V(\overrightarrow{\mathscr{A}\mathscr{B}}) + V(\overleftarrow{\mathscr{A}\mathscr{B}}) \subseteq V(\mathscr{A} \times \mathscr{B})$. Actually, this inclusion can be strengthened to an equality.

Theorem 5.12. $V(\overrightarrow{\mathscr{A}\mathscr{B}})+V(\overleftarrow{\mathscr{A}\mathscr{B}})=V(\mathscr{A}\times\mathscr{B})$

Proof. By Lemma 5.11 and Theorem 5.10, we have

$$\begin{aligned}
&\dim(V(\overrightarrow{\mathscr{A}\mathscr{B}})+V(\overleftarrow{\mathscr{A}\mathscr{B}}))\\
&\quad=\dim(V(\overrightarrow{\mathscr{A}\mathscr{B}}))+\dim(V(\overleftarrow{\mathscr{A}\mathscr{B}}))-\dim(V(\overrightarrow{\mathscr{A}\mathscr{B}})\cap V(\overleftarrow{\mathscr{A}\mathscr{B}}))\\
&\quad=\dim(V(\overrightarrow{\mathscr{A}\mathscr{B}}))+\dim(V(\overleftarrow{\mathscr{A}\mathscr{B}}))-\dim(V(\mathscr{A}\mathscr{B}))\\
&\quad=\dim(V(\mathscr{A}\times\mathscr{B}))
\end{aligned}$$

The theorem now follows from the fact that

$$V(\overrightarrow{\mathscr{A}\mathscr{B}})+V(\overleftarrow{\mathscr{A}\mathscr{B}})\subseteq V(\mathscr{A}\times\mathscr{B})$$

Corollary 5.13. Let $\omega\in\mathbb{R}^{XY}$. Then $\omega\in\Omega(\mathscr{A}\times\mathscr{B})$ if and only if $\omega(xy)\geq 0$ for all $xy\in XY$ and there exist $\vec{\omega}\in\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$, $\overleftarrow{\omega}\in\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$, and $t\in\mathbb{R}$ such that $\omega=t\vec{\omega}+(1-t)\overleftarrow{\omega}$.

Proof. It is clear that any $\omega\in\mathbb{R}^{XY}$ that satisfies the given conditions is a probability weight on $\mathscr{A}\times\mathscr{B}$. Conversely, let $\omega\in\Omega(\mathscr{A}\times\mathscr{B})$. Then, $\omega(xy)\geq 0$ holds for all $xy\in XY$. Furthermore, by Theorem 5.12, we can write $\omega=\nu+\mu$ with $\nu\in V(\overrightarrow{\mathscr{A}\mathscr{B}})$, $\mu\in V(\overleftarrow{\mathscr{A}\mathscr{B}})$. Select a strictly positive probability weight $\alpha\in\Omega(\mathscr{A}\mathscr{B})$, noting that α is automatically a strictly positive probability weight in both $\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ and $\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$. By Lemma 5.3, there exist $p, q>0$ and there exists $\vec{\eta}\in\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$ such that $\nu=p\alpha-q\vec{\eta}$. Likewise, there exist $u, v>0$ and there exists $\overleftarrow{\omega}\in\Omega(\overleftarrow{\mathscr{A}\mathscr{B}})$ such that $\mu=u\alpha-v\overleftarrow{\omega}$. It follows that

$$\omega=(p+u)\alpha-q\vec{\eta}-v\overleftarrow{\omega}$$

Let $E\in\mathscr{A}$, $F\in\mathscr{B}$, and sum both sides of the latter equation over all $xy\in EF$ to conclude that $1=p+u-q-v$. Hence,

$$\omega=(1+q+v)\alpha-q\vec{\eta}-v\overleftarrow{\omega}$$

Now define

$$\vec{\omega}=\frac{1+q+v}{1+v}\alpha-\frac{q}{1+v}\vec{\eta}=\frac{1}{1+v}\omega+\frac{v}{1+v}\overleftarrow{\omega}$$

noting that, on the one hand, $\vec{\omega}\in V(\overrightarrow{\mathscr{A}\mathscr{B}})$, and, on the other hand, $\vec{\omega}\in\Omega(\mathscr{A}\times\mathscr{B})$. Because $\vec{\omega}\in\Omega(\mathscr{A}\times\mathscr{B})$, it follows that $\vec{\omega}(xy)\geq 0$ for all $xy\in XY$. Also, $\vec{\omega}(xy)$ sums to 1 as we run xy through any operation $EF\in\mathscr{A}\times\mathscr{B}$. Since $EF\in\overrightarrow{\mathscr{A}\mathscr{B}}$, it follows from the fact that $\vec{\omega}\in V(\overrightarrow{\mathscr{A}\mathscr{B}})$ that $\vec{\omega}\in\Omega(\overrightarrow{\mathscr{A}\mathscr{B}})$. Furthermore, with $t=1+v$, we have $t\vec{\omega}=\omega+v\overleftarrow{\omega}$, so that

$$\omega=t\vec{\omega}+(-v)\overleftarrow{\omega}=t\vec{\omega}+(1-t)\overleftarrow{\omega}$$

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